

# Eigenvalues of the Laplacian on an elliptic domain

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## Abstract

The importance of eigenvalue problems concerning the Laplacian is well documented in classical and modern literature. Finding the eigenvalues for various geometries of the domains has posed many challenges which include infinite systems of algebraic equations, asymptotic methods, integral equations etc. In this paper, we present a comprehensive account of the general solutions to Helmholtz's equations (defined on simply connected regions) using complex variable techniques. We consider boundaries of the form  $z\bar{z} = f(z \pm \bar{z})$  or its inverse  $z \pm \bar{z} = g(z\bar{z})$ . To illustrate the theory, we reduce the problem on elliptic domains to equivalent linear infinite algebraic systems, where the coefficients of the infinite matrix are known polynomials of the eigenvalues. We compute truncations of the infinite system for numerical values. These values are compared to approximate values and some inequalities available in literature.

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## 1. Introduction

The eigenvalue problem of the Laplacian is represented by the Helmholtz equation, Telegraph equation or the equation of the vibrating membrane and is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + ku = 0, \quad \text{in } D, \quad (1a)$$

$$u = 0, \quad \text{on } \partial D, \quad (1b)$$

where  $D$  is a plane region bounded by smooth curve  $\partial D$ . The eigenvalue  $k_n$  and corresponding eigenfunctions  $u_n$  describe the natural modes of vibration of a membrane. According to the maximum principle,  $k_n$  must be positive for a nontrivial solution to exist. Also the eigenvalues  $k_n, n = 1, 2, \dots$  are ordered satisfying

$$0 < k_1 < k_2 < \dots < k_n < \dots$$

These eigenvalue problems are of fundamental importance in various fields including vibration systems, antenna analysis and quantum mechanics. The research has been extensive and is associated with hyperbolic equations via

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Fourier Transforms with respect to time, demonstrated by Hettich, Haaren, Ries and Still [1]. Keller and Rubinow [2, 3], introduced some notable works which include using WKB expansion to estimate the eigenvalues. With the elliptic domain, the Helmholtz's equation becomes separable into two well-known second order linear ordinary differential equations of the form

$$y'' + (a - 2q \cos 2x)y = 0 \quad (\text{Mathieu equation})$$

$$y'' - (a - 2q \cosh 2x)y = 0 \quad (\text{Modified Mathieu equation}).$$

The two equations are then numerically solved for various periodic boundary conditions yielding various odd and even solutions, which were collected in “Hand of Mathematical Functions” by Abramovich and Stegun [4]. For details and for visualization of the special eigenmodes we refer to the paper by Chen, Morris and Zhou [5]. The evaluations of the approximate eigenvalues complete with an error analysis for the Mathieu equations was demonstrated by Shivakumar, Rudraiah, and Williams [6]. An excellent account of asymptotic solutions can be found in [2] by Keller and Rubinow. Numerical approximations by using semi-infinite optimization is also given by Hettich, Haaren, Ries and Still [1]. In this paper we use complex variable techniques to write down the general solution to the Helmholtz equation as given by Vekua in [7] and then match analytically the boundary condition. In Section 2, we introduce the complex variables  $z = x + iy$ ,  $\bar{z} = x - iy$  and derive general solutions to the Helmholtz equation for simply connected regions. Section 3 describes some relationships for an elliptic boundary. In Section 4, we illustrate the theory by application to an elliptic region. In Section 5, the resulting infinite system of equations with coefficients which are known polynomials of the eigenvalues is truncated and numerical values are calculated. These values are compared with known approximate values (derived from asymptotic methods, etc.) in the article by Fox, Henrici and Moler [8]. Further, the inequalities derived in Kuttler and Sigillito's article [9] are also compared with the calculated numerical values. For doubly connected region, the behavior of the minimum eigenvalues of the Dirichlet Laplacian is discussed by Ramm and Shivakumar in [10]. Some results on doubly connected regions can be found in Wu's thesis [11]. Section 6 gives conclusion and indication for future work including doubly connected regions and eigenvalue problems for biharmonic equations, etc.

## 2. General solutions

We introduce the complex variables  $z = x + iy$  and  $\bar{z} = x - iy$ , and rewrite the two-dimensional problem (1a), using  $k = \lambda^2$ , in the complex form

$$\frac{\partial^2 u}{\partial z \partial \bar{z}} + \frac{\lambda^2}{4} u = 0, \quad \text{in } D, \quad (2a)$$

$$u = 0, \quad \text{on } C, \quad (2b)$$

where we use the same notation  $u(z, \bar{z})$  to denote the real value function of complex variables, and the same  $D$  and  $C$  to denote the domain and its boundary in the complex plane.

The general solution to (2a) in any simply connected domain can be expressed by [7, p. 58] demonstrated by Vekua

$$u = \left\{ f_0(z) - \int_0^z f_0(t) \frac{\partial}{\partial t} J_0 \left( \lambda \sqrt{\bar{z}(z-t)} \right) dt \right\} + \text{conjugate}, \quad (3)$$

where  $f_0(z)$  is an arbitrary holomorphic function which can be expressed as

$$f_0(z) = \sum_{n=0}^{\infty} a_n z^n, \quad (4)$$

$J_0$  in (3) is the Bessel's function of the first kind of order 0, which is given by a series,

$$J_0 \left( \lambda \sqrt{\bar{z}(z-t)} \right) = \sum_{k=0}^{\infty} \left( -\frac{\lambda^2}{4} \right)^k \frac{\bar{z}^k (z-t)^k}{k! k!}. \quad (5)$$

In fact, (3) can also be derived by using

$$u = \sum_{n=0}^{\infty} \bar{z}^n f_n(z) + \text{conjugate}, \quad (6)$$

yielding

$$f_n = \left(-\frac{\lambda^2}{4}\right)^n \frac{1}{n!} \frac{1}{(n-1)!} \int_0^z f_0(t)(z-t)^{n-1} dt \quad (7)$$

which will be used in evaluating  $u$  numerically in the region by truncating the series suitably in (6). Further, (6) can be used in finding general solutions of the Helmholtz equations in doubly connected region and also, in solving biharmonic equations.

On substituting (4) and (5) in (3), we can simplify the solution as,

$$\begin{aligned} u &= \left[ a_0 J_0(\lambda \sqrt{z\bar{z}}) + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \left(-\frac{\lambda^2}{4}\right)^k \frac{n a_n \bar{z}^k}{k! k!} \int_0^z t^{n-1} (z-t)^k dt \right] + \text{conjugate} \\ &= 2a_0 J_0(\lambda \sqrt{z\bar{z}}) + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \left(-\frac{\lambda^2}{4}\right)^k \frac{n a_n (z\bar{z})^k}{k! k!} B(n, k+1) (z^n + \bar{z}^n), \end{aligned} \quad (8)$$

where,  $B(n, k+1)$  denotes the Beta function with integer parameters  $n$  and  $k+1$ , and is given by

$$B(n, k+1) = \frac{(n-1)!k!}{(n+k)!}.$$

For simplicity, we denote a part of the constant coefficients by

$$\left(-\frac{\lambda^2}{4}\right)^k \frac{n}{k!k!} B(n, k+1) = A_{n,k}, \quad \begin{array}{l} n = 1, 2, \dots, \infty, \\ k = 0, 1, 2, \dots, \infty. \end{array} \quad (9)$$

We will rewrite (8) to give,

$$u = 2a_0 J_0(\lambda \sqrt{z\bar{z}}) + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} A_{n,k} (z^n + \bar{z}^n) a_n (z\bar{z})^k. \quad (10)$$

Eq. (10) gives the general solution of the Helmholtz equation on an arbitrary simply connected domain. Alternately, we can express  $z^n + \bar{z}^n$  in terms of  $z + \bar{z}$  and  $z\bar{z}$  and its powers. This is accomplished as follows.

Let  $z = re^{i\theta}$ , we have by the identity given in [12] created by Gradshteyn and Ryzhik,

$$\cos(n\theta) = 2^{n-1} \cos^n \theta - 2^{n-3} \frac{n}{1} \cos^{n-2} \theta + 2^{n-5} \frac{n}{2} \binom{n-3}{1} \cos^{n-4} \theta \dots,$$

and when  $n$  is even, we can rewrite

$$\begin{aligned} z^n + \bar{z}^n &= 2r^n \cos(n\theta) \\ &= r^n \left[ (2 \cos \theta)^n + \sum_{m=1}^{n/2} (-1)^m \frac{n}{m} \binom{n-m-1}{m-1} (2 \cos \theta)^{n-2m} \right] \\ &= (z + \bar{z})^n + \sum_{m=1}^{n/2} (-1)^m \frac{n}{m} \binom{n-m-1}{m-1} (z + \bar{z})^{n-2m} (z\bar{z})^m. \end{aligned} \quad (11)$$

In particular,

$$z^{2n} + \bar{z}^{2n} = 2r^{2n} \cos(2n\theta)$$

$$\begin{aligned}
&= r^{2n} \left[ (2 \cos \theta)^{2n} + \sum_{m=1}^n (-1)^m \frac{2n}{m} \binom{2n-m-1}{m-1} (2 \cos \theta)^{2n-2m} \right] \\
&= (z + \bar{z})^{2n} + \sum_{m=1}^n (-1)^m \frac{2n}{m} \binom{2n-m-1}{m-1} (z + \bar{z})^{2n-2m} (z\bar{z})^m.
\end{aligned} \tag{12}$$

Alternatively, Eq. (10) with Eq. (11) can be written as

$$u = 2a_0 J_0(\lambda \sqrt{z\bar{z}}) + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} A_{n,k} \left( (z + \bar{z})^n + \sum_{m=1}^{n/2} (-1)^m \frac{n}{m} \binom{n-m-1}{m-1} (z + \bar{z})^{n-2m} (z\bar{z})^m a_n \right) (z\bar{z})^k,$$

demonstrating that the general solution of (2a) without boundary conditions can be expressed in terms of power of  $z\bar{z}$  and  $(z + \bar{z})$ . A similar solution can be obtained by using (12).

### 3. Elliptic boundary

In our case, we consider the domain bounded by the ellipse

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1,$$

which can be expressed correspondingly in the complex plane by

$$(z + \bar{z})^2 = a + bz\bar{z}, \tag{13}$$

where  $a = \frac{4\alpha^2\beta^2}{\beta^2 - \alpha^2}$ ,  $b = \frac{4\alpha^2}{\alpha^2 - \beta^2}$ .

In (13),  $(z + \bar{z})^2$  is expressed in terms of  $z\bar{z}$ , yielding after substitution for  $z^{2n} + \bar{z}^{2n}$  and some simplifications,

$$\begin{aligned}
z^{2n} + \bar{z}^{2n} &= \sum_{l=0}^n \sum_{m=0}^l \frac{(-1)^m (2n-m-1)! 2n}{m! (2n-2m)!} \binom{n-m}{l-m} a^{n-l} b^{l-m} (z\bar{z})^l \\
&= \sum_{l=0}^n b_{l,n} (z\bar{z})^l,
\end{aligned} \tag{14}$$

where

$$b_{l,n} = \sum_{m=0}^l \frac{(-1)^m (2n-m-1)! 2n}{m! (2n-2m)!} \binom{n-m}{l-m} a^{n-l} b^{l-m}, \tag{15}$$

$l = 1, 2, \dots, n$  and  $b_{0,n} = a^n$ .

We can now substitute for  $z^{2n} + \bar{z}^{2n}$  in terms of  $z\bar{z}$  in (10) yielding,

$$\begin{aligned}
u &= 2a_0 J_0(\lambda \sqrt{z\bar{z}}) + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} A_{2n,k} (z\bar{z})^k \sum_{l=0}^n b_{l,n} (z\bar{z})^l a_n \\
&= 2a_0 J_0(\lambda \sqrt{z\bar{z}}) + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \sum_{l=0}^n A_{2n,k} b_{l,n} (z\bar{z})^{l+k} a_n \\
&= 2a_0 J_0(\lambda \sqrt{z\bar{z}}) + \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} A_{2n,k} b_{0,n} (z\bar{z})^k a_n + \sum_{k=0}^{\infty} \sum_{l=1}^{\infty} \left[ \sum_{n=l}^{\infty} A_{2n,k} b_{l,n} \right] (z\bar{z})^{l+k} a_n \\
&= 2a_0 J_0(\lambda \sqrt{z\bar{z}}) + \sum_{k=0}^{\infty} \sum_{n=1}^{\infty} A_{2n,k} b_{0,n} (z\bar{z})^k a_n + \sum_{k=1}^{\infty} \sum_{l=1}^k \left[ \sum_{n=l}^{\infty} A_{2n,k-l} b_{l,n} \right] (z\bar{z})^k a_n.
\end{aligned}$$

Substituting now the series for Bessel's function for  $J_0$  from (5), we can finally express  $u(z, \bar{z})$  in terms of a power series in  $(z\bar{z})$  given by

$$u = 2a_0 + \sum_{n=1}^{\infty} A_{2n,0} b_{0,n} a_n + \sum_{k=1}^{\infty} \left( -\frac{\lambda^2}{4} \right)^k \frac{2a_0}{k!k!} (z\bar{z})^k + \sum_{k=1}^{\infty} \left[ \sum_{n=1}^k \left( A_{2n,k} b_{0,n} + \sum_{l=1}^n A_{2n,k-l} b_{l,n} \right) a_n \right] (z\bar{z})^k \\ + \sum_{k=1}^{\infty} \left[ \sum_{n=k+1}^{\infty} \left( A_{2n,k} b_{0,n} + \sum_{l=1}^k A_{2n,k-l} b_{l,n} \right) a_n \right] (z\bar{z})^k, \quad (16)$$

where  $A_{2n,k}$  and  $b_{l,n}$  are given by (9) and (15) respectively. Now Eq. (16) gives the general solution to (1a) in the elliptic domain represented by (13).

#### 4. Boundary value problem

With the homogeneous boundary condition  $u = 0$  on the bounding ellipse, we need to solve for  $\lambda$ 's and  $a_n$ 's by setting  $u = 0$  in Eq. (16). This implies equating powers  $(z\bar{z})^k$  to zero for  $k = 0, 1, 2, \dots$  in (16).

Equating the constant term in (16) to zero, we get

$$2a_0 + \sum_{n=1}^{\infty} A_{2n,0} b_{0,n} a_n = 0, \quad (17)$$

while equating the coefficients of  $(z\bar{z})^k$  ( $k \neq 0$ ) to zero, yields

$$\left( -\frac{\lambda^2}{4} \right)^k \frac{2a_0}{k!k!} + \sum_{n=1}^k \left( A_{2n,k} b_{0,n} + \sum_{l=1}^n A_{2n,k-l} b_{l,n} \right) a_n + \sum_{n=k+1}^{\infty} \left( A_{2n,k} b_{0,n} + \sum_{l=1}^k A_{2n,k-l} b_{l,n} \right) a_n = 0, \\ k = 1, 2, \dots \quad (18)$$

Using the expressions for coefficients in (9) and (15) and letting  $\frac{\lambda^2}{4} = T$  for convenience, Eqs. (17) and (18) become

$$2a_0 + \sum_{n=1}^{\infty} a_n a^n = 0, \quad (19)$$

$$\frac{2a_0(-T)^k}{k!k!} + \sum_{n=1}^k \left( \frac{(2n)!(-T)^k a^n}{k!(2n+k)!} + \sum_{l=1}^n \frac{(2n)!(-T)^{k-l} b_{l,n}}{(k-l)!(2n+k-l)!} \right) a_n \\ + \sum_{n=k+1}^{\infty} \left( \frac{(2n)!(-T)^k a^n}{k!(2n+k)!} + \sum_{l=1}^k \frac{(2n)!(-T)^{k-l} b_{l,n}}{(k-l)!(2n+k-l)!} \right) a_n = 0, \quad k = 1, 2, \dots \quad (20)$$

Eqs. (19) and (20) yield an infinite system of linear equations with variables  $a_0, a_1, \dots$ . We can eliminate  $a_0$  and from Eq. (20), using (19) to derive the infinite system in the matrix form

$$D\vec{a} = 0, \quad \vec{a} = \{a_1, a_2, \dots\}, \quad (21)$$

where  $D = (d_{k,n})$  is an infinite matrix with its  $(k, n)$ th entry given by,

$$d_{k,n} = \begin{cases} \sum_{l=1}^n \frac{(-T)^{k-l} (2n)!}{(2n+k-l)!(k-l)!} b_{l,n} + \frac{a^n (-T)^k}{k!} \left( \frac{(2n)!}{(2n+k)!} - \frac{1}{k!} \right) & n \leq k \\ \sum_{l=1}^k \frac{(-T)^{k-l} (2n)!}{(2n+k-l)!(k-l)!} b_{l,n} + \frac{a^n (-T)^k}{k!} \left( \frac{(2n)!}{(2n+k)!} - \frac{1}{k!} \right) & n > k, \end{cases} \quad (22)$$

$k = 1, 2, \dots, \infty$  and  $n = 1, 2, \dots, \infty$ .

## 5. Numerical solution of the system (22)

To get numerical values for the eigenvalues of the boundary value problem (2a), we truncate the infinite matrix in (22) to give  $D^{(n)}$  an  $n \times n$  matrix with  $n$  equations and  $n$  unknowns  $a_1, a_2, \dots, a_n$ . We now find the zeros of  $\det D^{(n)}$ . Notice that, in the expression of  $d_{k,n}$  in (22), we can factor  $(-T)^k$  in each row of the matrix, and the vanishing of the determinant of the matrix  $D^{(n)}$  can be expressed as

$$\det D^{(n)} = \det E^{(n)} = 0,$$

where the  $(k, n)$ th component of matrix  $E^{(n)} = (e_{k,n})$  is given by,

$$e_{k,n} = \begin{cases} \sum_{l=1}^n \frac{(-T)^{-l}(2n)!}{(2n+k-l)!(k-l)!} b_{l,n} + \frac{a^n}{k!} \left( \frac{(2n)!}{(2n+k)!} - \frac{1}{k!} \right) & n \leq k, \\ \sum_{l=1}^k \frac{(-T)^{-l}(2n)!}{(2n+k-l)!(k-l)!} b_{l,n} + \frac{a^n}{k!} \left( \frac{(2n)!}{(2n+k)!} - \frac{1}{k!} \right) & n > k, \end{cases} \quad (23)$$

$k, n = 1, 2, 3, \dots$

Setting  $\det E^{(n)} = 0$ , we get a polynomial equation of degree  $\frac{n(n+1)}{2}$  in  $T$  yielding  $\frac{n(n+1)}{2}$  zeros. The zeros will consist of real numbers and complex conjugates. There will be inherent errors due to truncation of the infinite systems and due to computations.

When  $n = 2$ , the explicit form of  $E^{(2)}$  is given by,

$$E^{(2)} = \begin{bmatrix} -\frac{2}{3}a + \frac{2-b}{T}, & -\frac{4a^2}{5} + \frac{4a-2ab}{T} \\ -\frac{5a}{24} + \frac{2-b}{3T}, & -\frac{7a^2}{30} + \frac{4a-2ab}{5T} + \frac{b^2-4b+2}{T^2} \end{bmatrix}$$

giving, for  $\det E^{(2)} = 0$ , the polynomial equation of degree 3 in  $T$  as

$$2a^3T^3 + 33a^2T^2b - 66a^2T^2 + 168aTb^2 - 672abT + 432aT + 180b^3 - 1080b^2 + 1800b - 720 = 0. \quad (24)$$

When  $n = 3$ , the explicit form of  $E^{(3)}$  is given by

$$E^{(3)} = \begin{bmatrix} -\frac{2}{3}a + \frac{2-b}{T} & -\frac{4a^2}{5} + \frac{4a-2ab}{T} & -\frac{6a^3}{7} + \frac{6a^2-3a^2b}{T} \\ -\frac{5a}{24} + \frac{2-b}{3T} & -\frac{7a^2}{30} + \frac{4a-2ab}{5T} + \frac{b^2-4b+2}{T^2} & -\frac{27a^3}{112} + \frac{6a^2-3a^2b}{7T} + \frac{3ab^2-12ab+9a}{T^2} \\ -\frac{a}{40} + \frac{2-b}{24T} & -\frac{17a^2}{630} + \frac{4a-2ab}{60T} + \frac{b^2-4b+2}{5T^2} & -\frac{83a^3}{3024} + \frac{6a^2-3a^2b}{112T} + \frac{3ab^2-12ab+9a}{7T^2} + \frac{-b^3+6b^2-9b+2}{T^3} \end{bmatrix}$$

giving, for  $\det E^{(3)} = 0$ , the polynomial of degree 6 in  $T$ .

For purposes of comparison with some known results, we follow the values chosen in Section 3.1 of [5] by Chen, Morris and Zhou, where  $\alpha = \cosh 2 = 3.762195691$ ,  $\beta = \sinh 2 = 3.626860408$  and  $c = 2$  where  $kc/2$  corresponds to our  $\lambda = \sqrt{4T}$ . Note that  $a = -744.7395806$ ;  $b = 52.61646567$  are given by (13) and  $2 \times 2$  truncation of the infinite matrix is given by,

$$E^{(2)} = \begin{bmatrix} 496.493 - \frac{54.616}{T}, & -443\,709.635 + \frac{81\,350.088}{T} \\ 155.154 - \frac{18.205}{T}, & -129\,415.310 + \frac{16\,270.0175}{T} + \frac{2980.958}{T^2} \end{bmatrix}$$

and letting  $\det E^{(2)} = 0$  yields a polynomial of degree 3 in  $T$  and is given by

$$4589\,557.35T^3 - 5553\,591.10T^2 + 2072\,432.337T - 162\,809.4079 = 0.$$

Table 1  
Values of  $\lambda$ 

n	0.65123129	1.49784709	2.35503473	3.21749843	4.08203626	4.94732655
2	0.65134447	N.A	N.A	N.A	N.A	N.A
3	0.65123159	N.A	N.A	N.A	N.A	N.A
4	0.65123129	1.49618630	N.A	N.A	N.A	N.A
5	0.65123129	1.49781766	2.38675835	N.A	N.A	N.A.
7	0.65123129	1.49784709	2.35533376	N.A.	N.A	N.A
10	0.65123129	1.49784709	2.35503473	3.21744280	N.A.	N.A.
15	0.65123129	1.49784709	2.35503473	3.21749356	4.08204066	4.94145456

The zeros for  $T$  are given by 0.1060624033 and  $0.5519934964 \pm i0.1725268855$ . Similarly,  $\det E^{(3)} = 0$  yields a polynomial of degree 6, given by

$$286.58542T^6 - 1481.70752T^5 + 3093.92157T^4 - 3273.942769T^3 + 1722.787033T^2 - 399.2807381T + 26.49801122 = 0.$$

The roots are 0.1060256457,  $0.5216052059 \pm i0.02470374531$ ,  $1.125865529 \pm i0.7348683331$  and 1.769244136. It is easy to see that  $\det E^{(3)} = 0$  yielded two real roots for  $\lambda = \sqrt{4T}$ .

In Table 1, the top row gives the values of  $\lambda$  according to [5] by Chen, Morris and Zhou, and  $n$  is the size of the truncated matrix. The numerical values we obtained from our method compares very well with the results in [5] by Chen, Morris and Zhou. It gives very good estimations to several of the first eigenvalues by solving relatively small systems. The larger the truncation matrix is, the more precise the numerical values will be. Note that in the article by Chen, Morris and Zhou [5], the approximate values were found by asymptotic methods without a precise error analysis. To ensure the required degree of accuracy in the computation of the eigenvalues of the truncated matrices, in Table 1, we have used the method of bisection.

For comparison of numerical values obtained by truncation, we now state the following known result involving inequalities and approximation for the first eigenvalue  $\lambda_1$ .

The Faber–Krahn inequality states that the circular region has the smallest  $\lambda_1^2$  of all regions with the same area. The two-dimensional Faber–Krahn inequality can be expressed as

$$\lambda_1^2 \geq \frac{\pi}{A} j_{0,1}^2, \quad (25)$$

where  $A$  is the area of the region and  $j_{0,1} = 2.4048$  is the first zero of the Bessel function  $J_0$ . We can compare our numerical result of the first eigenvalue from Table 1

$$4T = \lambda_1^2 = (0.65134467)^2 \approx 0.42410 \geq \frac{\pi}{A} j_{0,1}^2 = 0.423833. \quad (26)$$

We now compare our numerical results with those given in [8, p. 96] introduced by Fox, Henrici and Moler. The sizes of the matrices used to calculate the first eigenvalue are as low as  $5 \times 5$ .

In Fox, Henrici and Moler's book [8], the numerical values were obtained by collocation using interpolation methods without an error analysis.

From Table 2, we can see that the error  $\epsilon$  of the first eigenvalue satisfies that

$$\epsilon \leq 10^{-5}.$$

We can also obtain higher accuracy for the eigenvalues by truncating the infinite matrices at larger  $n$ . From the last column of Table 2, we see that the Faber–Krahn inequality is also satisfied.

## 6. Conclusion and future work

In this paper, we expressed the solutions to (2a) and (2b) where (2b) is an ellipse given by (13) in terms of an equivalent infinite system given by (21). The coefficients in (21) are known polynomials of the eigenvalues which makes it a challenge to develop an error analysis.

Table 2

First eigenvalue of ellipses (semiminor axis = 1.0) and  $A$  is the area of the ellipse

Semimajor axis	Lower bound [8] Upper bound	First eigenvalue from (22)	$\frac{\pi}{A} j_{0,1}^2$
1.1	5.2800383 $\frac{3}{4}$	5.280038336	5.257441784
1.2	4.8952213 $\frac{3}{5}$	4.895221340	4.819321636
1.4	4.353418 $\frac{36}{69}$	4.353418376	4.130847116
1.6	3.9968064 $\frac{7}{9}$	3.996706485	3.614491227
1.8	3.7481592 $\frac{2}{3}$	3.074815922	3.212881091
2	3.566726 $\frac{58}{62}$	3.566726605	2.891592981
3	3.108128 $\frac{66}{75}$	3.018128730	1.927728654
4	2.92025 $\frac{080}{120}$	2.920251099	1.445796491

The approach using complex variables to solve the eigenvalue problem for the Laplacian has many advantages. Firstly, the solutions to the Helmholtz equations are independent of the boundary conditions. Secondly, the method can easily be applied to many two-dimensional problems defined on analytic arbitrary convex domains, where we can express the boundary in the form  $z \pm \bar{z} = f(z\bar{z})$  or  $z\bar{z} = g(z \pm \bar{z})$ . Some examples include a circular disk bounded by  $z\bar{z} = r^2$ , a rectangular domain bounded by  $[(z + \bar{z})^2 - a^2][(z + \bar{z})^2 - 2z\bar{z} + b^2] = 0$ . The present method has many applications when the regions are doubly connected. An account of general solutions for the doubly connected regions is given in Wu's thesis [11]. For a result concerning inequalities for the minimum eigenvalue of the Laplacian in an annulus see [10] by Ramm and Shivakumar. Further work involves justification of the truncations of the infinite matrix complete with error analysis.

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